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On Certain Conics Connected with Trinodal Quartics.

BY A. B. BASSET.

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1. It is well known that three conics can be described which respectively pass, (i), through the six points in which the nodal tangents intersect the quartic, (ii), through the six points of contact of the tangents drawn from the nodes, (iii), through the six points of inflexion; also, that each of the three conics intersect the quartic at two points S, S', which will be called the S points. A somewhat lengthy proof of the first theorem is given in §194 of my book on "Cubic and Quartic Curves;" and concise proofs of the three theorems, together with the equations of the three conics, appear to be a *desideratum*.

There is also a fourth conic which passes through the six Q points of a trinodal quartic. The equation of this conic will be found, and it will be shown to pass through two points T, T', which will be called the T points, which are the two remaining points in which the line SS' cuts the quartic. The T points are also points of importance in the theory of trinodal quartics.

Let the equation of the quartic be

$$\beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2 + 2\alpha\beta\gamma\,(l\alpha + m\beta + n\gamma) = 0, \tag{1}$$

to which form every trinodal quartic can be reduced by projection. Let

$$\sigma = l\beta\gamma + m\gamma\alpha + n\alpha\beta, \tag{2}$$

$$\tau = \beta\gamma/l + \gamma\alpha/m + \alpha\beta/n, \tag{3}$$

$$u = k_1\alpha + k_2\beta + k_3\gamma, \tag{4}$$

where

$$k_1 = m/n + n/m - 2l, \tag{5}$$

with similar expressions for k_2, k_3. Then (1) may be written in the form

$$\sigma\tau - \alpha\beta\gamma u = 0. \tag{6}$$

Equation (6) shows that the line $u = 0$ intersects the quartic in four points, two of which, S, S', lie in the conic $\sigma = 0$, whilst the remaining two, T, T', lie in the conic $\tau = 0$. And if §194 of my book, $2l$, $2m$, $2n$ be written for l, m, n, it can easily be shown that the equation of the conic may be written in the form

$$\Delta\sigma + u(\alpha/l + \beta/m + \gamma/n) = 0, \tag{7}$$

where
$$\Delta = -4 - \frac{1}{l^2} - \frac{1}{m^2} - \frac{1}{n_2} + \frac{4(l^2+m^2+n^2)+1}{2lmn},$$

which shows that the conic (7), which passes through the six points in which the nodal tangents intersect the quartic, also passes through the S points.

2. *To find the equation of the conic which passes through the six points of contact of the tangents drawn from the nodes.*

Equation (1) may be written in the form

$$\alpha^2\{\beta^2+\gamma^2+2l\beta\gamma-(m\beta+n\gamma)^2\}+\{\alpha(m\beta+n\gamma)+\beta\gamma\}^2=0, \tag{8}$$

which shows that the equation of the tangents drawn from the node A is

$$\beta^2+\gamma^2+2l\beta\gamma-(m\beta+n\gamma)^2=0, \tag{9}$$

and that the points of contact lie in the conic

$$\alpha(m\beta+n\gamma)+\beta\gamma=0, \tag{10}$$

from which it is easily shown that the equation of the chord of contact of the tangents is

$$k_1\alpha+\left(\frac{1}{m}-m\right)\beta+\left(\frac{1}{n}-n\right)\gamma=0, \tag{11}$$

The equation
$$\sigma+u(L\alpha+M\beta+N\gamma)=0 \tag{12}$$

represents some conic passing through the S points; and we have to show that it is possible to determine L, M, N, so that (12) passes through the required six points. Combining (12) with (6), it follows that the remaining six points of intersection of (12) and (6) lie in the cubic

$$\tau(L\alpha+M\beta+N\gamma)+\alpha\beta\gamma=0. \tag{13}$$

Since (13) passes through A, B and C, it will intersect the conic (10) in three other points; and by substituting the value of $\alpha(m\beta+n\gamma)$ from (10) in (13), one of the three chords of intersection of (10) and (13) will be found to be

$$p_1(L\alpha+M\beta+N\gamma)+\alpha=0, \tag{14}$$

where
$$p_1=\frac{1}{l}-\frac{1}{mn}$$

with similar expressions for p_2, p_3. The conditions that (14) should be identical with (11) are that

$$\frac{Lp_1+1}{k_1}=\frac{Mmp_1}{1-m^2}=\frac{Nnp_1}{1-n^2}.$$

By dealing with the two pairs of tangents from B and C in a similar manner, we shall obtain two more sets of equations, one of which is

$$\frac{Mp_2+1}{k_2}=\frac{Nnp_2}{1-n^2}=\frac{Llp_2}{1-l^2}.$$

These six equations are not, however, independent, and will be found to uniquely determine the values of L, M, N which are given by

$$L\Delta_1=\frac{1}{l}-l,\text{ etc., etc.,}$$

$$\Delta_1=\frac{1+l^2+m^2+n^2}{lmn}-1-\frac{1}{l^2}-\frac{1}{m^2}-\frac{1}{n^2},$$

showing that the required conic is

$$\Delta_1\sigma+u\left\{\left(\frac{1}{l}-l\right)\alpha+\left(\frac{1}{m}-m\right)\beta+\left(\frac{1}{n}-n\right)\gamma\right\}=0, \tag{15}$$

which, therefore, passes through the S points.

A similar method can be employed to obtain the conic (7), in which the conic $2\alpha(m\beta+n\gamma)+\beta\gamma=0$ which passes through the points where the nodal tangents intersect the quartic, must be used in the place of (10).

3. *To find the equation of the conic passing through the six points of inflexion.*

The method of quadric inversion consists in writing for each coordinate its reciprocal; accordingly a straight line inverts into a conic circumscribing the triangle of reference and *vice versa;* a conic arbitrarily situated inverts into a trinodal quartic: whilst a cubic circumscribing the triangle of reference inverts into another cubic circumscribing the same triangle. Hence the quartic (1) inverts into the conic

$$S=\alpha^2+\beta^2+\gamma^2+2l\beta\gamma+2m\gamma\alpha+2n\alpha\beta=0 \tag{16}$$

and the six stationary tangents invert into six osculating conics which circumscribe the triangle of reference. Let $2U=dS/df$, $2V=dS/dg$, $2W=dS/dh$; then the conic

$$S+(\alpha U+\beta V+\gamma W)(\lambda\alpha+\mu\beta+\nu\gamma)=0 \tag{17}$$

represents a conic which osculates S at the point (f, g, h) on it; hence if (17) circumscribe the triangle of reference

$$\lambda U = \mu V = \nu W = -1,$$

accordingly since $(f, g, h,)$ lies in the line (λ, μ, ν), we obtain

$$\frac{f}{U} + \frac{g}{V} + \frac{h}{W} = 0, \tag{18}$$

which shows that the points of contact of the six conics lie in the cubics (18). Writing (α, β, γ) for (f, g, h) we obtain

$$U = \alpha + n\beta + m\gamma,$$
$$= \alpha p_1 + mnv_1,$$

where $$v = \alpha/l + \beta/m + \gamma/n,$$
$$p_1 = 1 - mn/l, \text{ etc., etc.},$$

whence $$VW = p_2p_3\beta\gamma + lv(m\beta + n\gamma + mn\alpha),$$

accordingly the cubic (18) becomes

$$\alpha\beta\gamma\,(p_2p_3 + p_3p_2 + p_1p_2) + lmnv\,(\alpha^2 + \beta^2 + \gamma^2 + 2\beta\gamma/l + 2\gamma\alpha/m + 2\alpha\beta/n) = 0.$$

Subtracting $lmn\,Sv$, and writing Δ_2 for $p_2p_3 + p_3p_1 + p_1p_2$ it follows that the cubic

$$\Delta_2\alpha\beta\gamma + 2lmn\left\{\left(\frac{1}{l} - l\right)\beta\gamma + \left(\frac{1}{m} - m\right)\gamma\alpha + \left(\frac{1}{n} - n\right)\alpha\beta\right\}v = 0, \tag{19}$$

circumscribes the triangle of reference and passes through the points of contact of the six osculating conics. Inverting (19), the cubic

$$\Delta_2\alpha\beta\gamma + 2lmn\varpi\left\{\left(\frac{1}{l} - l\right)\alpha + \left(\frac{1}{m} - m\right)\beta + \left(\frac{1}{n} - n\right)\gamma\right\} = 0,$$

passes through the nodes and the six points of inflexion of the quartic. Now this cubic is of the same form as the cubic (13), and we can therefore pass from it to the conic passing through the points of inflexion. Accordingly, the equation of the conic is

$$\Delta_2\sigma + 2lmn\left\{\left(\frac{1}{l} - l\right)\alpha + \left(\frac{1}{m} - m\right)\beta + \left(\frac{1}{n} - n\right)\gamma\right\}u = 0, \tag{20}$$

when $$\Delta_2 = 3 + l^2 + m^2 + n^2 - 2(m^2n^2 + n^2l^2 + l^2m^2)/lmn.$$

It is worthy of note that the conic passing through the points of contact of the tangents from the nodes and the conic passing through the points of inflexion intersect the conic $\sigma = 0$ in the same four points, viz., the S points and the points where σ is cut by the line

$$\left(\frac{1}{l} - l\right)\alpha + \left(\frac{1}{m} - m\right)\beta + \left(\frac{1}{n} - n\right)\gamma = 0.$$

4. *A cubic can be described through the six points where the stationary tangents intersect the quartic, which osculates the latter at the T points.*

To prove this theorem, we shall employ the parametric method in the form used by Mr. R. A. Roberts.* Let u, u'; v, v',; w, w' be the parameters of the nodes A, B and C; also let

$$p_1 = \frac{(u-v)(u-v')(u-w)(u-w')}{(u'-v)(u'-v')(u'-w)(u'-w')},$$

with similar expressions for p_2, p_3.

Then, if θ be the parameter of any point of inflexion, and x the parameter of the points where the corresponding stationary tangent cuts the quartic

$$\frac{(u-\theta)^3(u-x)}{(u'-\theta)^3(u'-x)} = p_1$$

with two similar equations. Whence

$$\Pi_1^6 \left(\frac{u-\theta_n}{u'-\theta_n}\right)^3 \left(\frac{u-x_n}{u-x_n}\right) = p_1^6,$$

but since the six points of inflexion lie on a conic passing through the S points it follows that if s, s' be the parameters of these points, and t, t' those of the T points

$$\frac{(u-s)(u-s')}{(u'-s)(u'-s')} \Pi_1^6 \left(\frac{u-\theta_n}{u'-\theta_n}\right) = p_1^2,$$

and

$$\frac{(u-s)(u-s')(u-t)(u-t')}{(u'-s)(u'-s')(u'-t)(u'-t')} = p_1,$$

whence

$$\frac{(u-t)^3(u-t')^3}{(u'-t)^3(u'-t')^3} \Pi_1^6 \left(\frac{u-x_n}{u'-x_n}\right) = p_1^3,$$

which proves the proposition.

5. Let the tangents at the node A of a quartic cut the curve in D, D'; and let Q, q be the two remaining points of intersection of the line DD' with the quartic; then the points Q, q have important properties with respect to the

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* Proc. Lond. Math. Soc., Vol. XVI, p. 44.

quartic which have been discussed by Mr. Westrop Roberts.* Now a trinodal quartic has a pair of Q points corresponding to each node, and all the properties which Mr. Roberts has proved for the Q points of a uninodal quartic are true for each pair of Q points of a trinodal quartic. In §194 of my book, I have shown that the equation of the line joining the points where the nodal tangents at A intersect the quartic is (since $2l$, $2m$, $2n$ are to be written for l, m, n),

$$2k_1\alpha + \beta/m + \gamma/n = 0. \tag{21}$$

Substituting the value of α from (21) in (1) we obtain

$$(\beta^2 + \gamma^2 + 2l\beta\gamma)\{(\beta/m + \gamma/n)^2 - 4k_1\beta\gamma\} = 0,$$

which shows that the equation of the lines AQ_1, Aq_1, is

$$(\beta/m + \gamma/n)^2 - 4k_1\beta\gamma = 0. \tag{22}$$

The equations of the lines BQ_2, Bq_2, CQ_3, Cq_3 can be written down by symmetry.

6. *A conic can be described which touches the six lines joining each node with its corresponding pair of Q points.*

The investigation of §192 of my book shows that if the equations of three pairs of straight lines are of the form

$$\begin{aligned} m\beta^2 + n\gamma^2 + \lambda\beta\gamma &= 0, \\ n\gamma^2 + l\alpha^2 + \mu\gamma\alpha &= 0, \\ l\alpha^2 + m\beta^2 + \nu\alpha\beta &= 0, \end{aligned}$$

a conic can be described touching them; and since (22) shows that the three pairs of lines in question are of this form, the theorem is proved.

7. *A conic can be described which passes through the nodes B and C, the two Q points corresponding to A, and the points of contact of the tangents from A; and its equation is*

$$k_1\alpha^2 + \alpha(\beta/m + \gamma/m) + \beta\gamma = 0. \tag{23}$$

This is a particular case of one of Mr. Roberts's theorems; but the proof leads to the equation of a conic which will be required in the next article. It may be verified as follows. From (21) and (22) we obtain

$$\alpha(\beta/m + \gamma/n) + 2\beta\gamma = 0, \tag{24}$$

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* Proc. Lond. Math. Soc., Vol. XXV, p. 151.

which is the equation of the conic passing through the nodes and the two Q points corresponding to A. Eliminating α between (23) and (24) we obtain (22), which shows that (23) passes through the Q points corresponding to A. Eliminating α between (23) and the conic (10), which passes through the points of contact of the tangents from the node, we obtain the equation of these tangents which shows that (23) passes through their points of contact.

8. *To prove that a conic can be described through the six Q points and the two T points.*

The equation

$$\tau + u(L\alpha + M\beta + N\gamma) = 0,$$

represents some conic passing through the T points. And if we proceed in the same way as in §2 making use of the conic (24) and the line (21) in the place of the conic (10) and the line (11) we shall find that the values of L, M, N are given by the equations

$$Ll = Mm = Nn = -\Delta_3^{-1}$$

when $\Delta_3 = 1 + 4(l^2 + m^2 + n^2) - 8lmn - 2(l^2m^2 + n^2l^2 + m^2n^2)/lmn.$

9. In §287 of my book, I have proved that the locus of the point of intersection of two tangents at the extremities of a chord through the node of a nodobicuspidal quartic is a nodal cubic which passes through the cusps of the quartic. A similar proposition is true in the case of a chord drawn through a cusp, but as the cusps of a limaçon are imaginary the method is not applicable. Let the equation of the quartic be

$$(\beta\gamma + \gamma\alpha + \alpha\beta)^2 = m^2\alpha^2\beta\gamma,$$

to which form any nodobicuspidal quartic can be reduced by projection. Now it is known that any point in the quartic can be expressed in terms of a parameter θ by means of the equation

$$\alpha = -\theta^2, \quad \beta = \theta^2 - m\theta + 1, \quad \gamma = \theta^2(\theta^2 - m\theta + 1). \tag{25}$$

The polar cubic of any point (ξ, η, ζ) is

$$\xi\{2S(\beta + \gamma) - 2m^2\alpha\beta\gamma\} + \eta\{2S(\gamma + \alpha) - m^2\alpha^2\gamma\}$$
$$+ \zeta\{2S(\alpha + \beta) - m^2\alpha^2\beta\} = 0, \tag{26}$$

where $S=\beta\gamma+\gamma\alpha+\alpha\beta$. Substituting the values of α, β, γ from (25) in (26), we obtain

$$2\theta^4(\xi+\eta)-m\theta^3(4\xi+\eta)+2\theta^2(m^2+2)\zeta-m\theta(4\xi+\zeta)+2(\xi+\zeta)=0. \tag{27}$$

This equation determines the parameters of the points of contact of the four tangents drawn from (ξ, η, ζ).

The equation of any line through the cusp B is $\gamma+(1-k)\alpha=0$, and therefore the parameters of the two points where it intersects the quartic are determined by

$$\theta^2-m\theta+k=0. \tag{28}$$

Hence, if θ_3, θ_4 be the roots of (28), the roots of (27), must be θ_1, θ_1, θ_3, θ_3, and since $\theta_3+\theta_4=m$, $\theta_3\theta_4=k$, we obtain from (27)

$$\begin{aligned}\theta_1+\theta_2+m&=\tfrac{1}{2}m(4\xi+\eta)/(\xi+\eta),\\ m(\theta_1+\theta_2)+\theta_1+\theta_2+k&=(m^2+2)\xi/(\xi+\eta),\\ k(\theta_1+\theta_2)+m\theta_1\theta_2&=\tfrac{1}{2}m(4\xi+\zeta)/(\xi+\eta),\\ k\theta_1\theta_2&=(\xi+\zeta)/(\xi+\eta).\end{aligned}$$

Eliminating θ_1, θ_2 and k, we obtain

$$(m^2\eta-\zeta)\{2\xi(6\eta-m^2\eta+\zeta)+m^2\eta^2+2\eta\zeta\}=18\eta^2(\xi+\zeta),$$

which is a cubic having a node at A and which passes through C.

FLEDBOROUGH HALL, HOLYPORT, BERKS, ENGLAND, *April* 21, 1903.